TWO-VELOCITY THEORY OF FLOW PAST SLENDER PROFILE

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We examine a hydrodynamic scheme for supersonic flow of a two-phase medium past a slender body. This theory has practical value in calculations of air-breathing jet engine inlets (when the gas contains liquid or solid particles); in calculations of the flight of bodies in dusty gas media; for pulp or sandstorm flow past obstacles; in several questions of atomized-propellant burning; and in the decomposition of the thermal protective spacecraft coatings. The linearized problem of two-phase (supersonic, barotropic) flow past bodies is solved using [1] as the basis.

The study showed that the problem does not have a solution if the surface of the given body forms streamlines of both phases. This fact is physically quite explainable since in the theory in question the pressure field is the same in both phases. Therefore, while one component flows about the body the other flow should not reach the body at all, i.e., the components separate and near the body there appears a region in which only the motion of one ("light" or "heavy") phase is significant.

We note that a model was proposed in [2] for two-fluid hydrodynamics of a fluidized bed in which the existence of a liquid-phase region near the wall is also shown. A light-fluid-phase region near the wall was noted in [3-7] in studies of nozzle particulate gas-particle flows, and so on.

The problem simplifies if the phase (component) interface is assumed and the corresponding profile is found (inverse problem). In the following we present an analytic solution of this (inverse) problem. The results obtained also make it possible to demonstrate some qualitative characteristics of the flow in question.

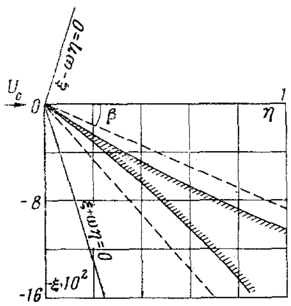

Fig. 1

We examine flow of a two-phase medium with initial velocity U_0 past a profile. We assume that the phases are mutually "nonwettable." Two flow types are possible: expansive and compressive. In the case of expansive flow there is complete separation of the light phase from the heavy phase above the surface of the profile. The former will flow about the body while the latter will travel in the two-phase flow region, whose boundary will be some streamline of the heavy phase, termed the phase interface (Fig. 1, dashed line). For compressive flow the motion pattern will be analogous to that for expansive flow except that the roles of the heavy and light phases are interchanged.

In describing the two-component system we usually start from the idea of the components as interpenetrating and interacting continuous media. The corresponding equations of motion are discussed in [1]. In the plane steady flow case they have the form

$$u_n \frac{\partial u_n}{\partial x} + v_n \frac{\partial u_n}{\partial y} = -\frac{1}{\rho_{ni}} \frac{\partial p}{\partial x} + \frac{1}{\rho_n} \sum_{j=1}^{2} K_{jn} (u_j - u_n)$$
$$u_n \frac{\partial v_n}{\partial x} + v_n \frac{\partial v_n}{\partial y} = -\frac{1}{\rho_{ni}} \frac{\partial p}{\partial y} + \frac{1}{\rho_n} \sum_{j=1}^{2} K_{jn} (v_j - v_n) \qquad (1)$$

The continuity equation is

$$\frac{\partial}{\partial x}(\rho_n u_n) + \frac{\partial}{\partial y}(\rho_n v_n) = 0 \qquad (2)$$

We supplement system (1) and (2) with the equations of state of the phases

$$p = f_n(\rho_{ni}) \qquad (3)$$

and the relation

$$\frac{\rho_1}{\rho_{1i}} + \frac{\rho_2}{\rho_{2i}} = 1 \qquad (n = 1,2) \qquad (4)$$

Here p is pressure, u_n and v_n are velocities, ρ_{ni} and ρ_n are the true and reduced densities of the n-th component, K_{jn} is the interaction function of the n-th component with the j-th component (in order to represent the solution in analytic form the function K_{jn} is assumed to be a constant quantity k).

We apply the small perturbation method to (1)–(4), i.e., we examine the linearized theory of flow past bodies. Then (1)–(4) in the case of irrotational potential flow with certain simplifications take the form

$$\frac{\rho_{10}}{\rho_0}[(M_1^2-1)\varphi_{1xx} - \varphi_{1yy}] + \frac{\rho_{20}}{\rho_{00}}[(M_2^2-1)\varphi_{2xx} - \varphi_{2yy}]$$

$$= -\frac{k}{U_0}\left(\frac{M_1^2}{\rho_0} - \frac{M_2^2}{\rho_{00}}\right)(\varphi_{1x} - \varphi_{2x})$$

$$\left(M_1 = \frac{U_0}{a_1}, \quad M_2 = \frac{U_0}{a_2}\right) \qquad (5)$$

$$\frac{\rho_{20}}{\rho_{00}}\rho_{10}\varphi_{1x} - \frac{\rho_{10}}{\rho_0}\rho_{20}\varphi_{2x} = -\frac{k}{U_0}(\varphi_1 - \varphi_2) \qquad (6)$$

Here φ_1 and φ_2 are the velocity potentials; M_1 and M_2 are the Mach numbers; ρ_0, ρ_{00} and ρ_{10}, ρ_{20} are the initial values of the true and reduced densities of the media. Let us now formulate the boundary conditions.

The equations of the two-component system are not valid near the wall. As we indicated in the beginning of the article, for expansive flow the light medium occupies the region near the wall. Therefore the limit of the action of the heavy phase will be the phase interface, on which the boundary conditions must be specified.

Let the phase interface be given by a straight line forming the angle β with the x-axis. It is obvious that this line is a streamline of the more dense (heavy) medium, i.e.,

$$\varphi_{1y}(x, y) = -U_0\beta \quad \text{for } y = \beta x \qquad (7)$$

and the light medium plays the dominant role in the region near the wall; therefore the condition of single-phase flow past the profile is satisfied as the surface of the body is approached. Additionally, the velocities of the two-phase system at infinity are bounded and on the characteristic lines

$$\varphi_1 = \varphi_2 = 0 \qquad (8)$$

Thus, the model under discussion makes it possible to pose the following boundary conditions: on the interphase surface, continuity of the normal and tangential components of the velocity of the light (or heavy) phase and the condition of flow of the heavy (or light) phase past this surface; at the solid boundary, the condition of flow of the light (or heavy) phase medium past the boundary. We apply the Laplace transform [8] to system (5), (6). Then the solutions of (5) and (6), satisfying boundary conditions (7) and (8), have the form

$$\varphi_1(x, y) = U_0\beta\omega e^{-cy/\omega}\left[w(t^*) + \int_0^{t^*}\Psi(t^* - \tau, y)w(\tau)\,d\tau\right]$$

$$\varphi_2(x, y) = U_0\beta\omega\frac{\rho_0}{\rho_{00}}e^{-cy/\omega}\Big\{w(t^*) + \int_0^{t^*}\Psi(t^* - \tau, y)w(\tau)\,d\tau$$

$$-a\left[\int_0^{t^*}e^{-m(t^*-\tau)}w(\tau)\,d\tau + \int_0^{t^*}\Psi(t^* - \tau, y)\int_0^{\tau}e^{-m(\tau-\tau_1)}w(\tau_1)\,d\tau_1 d\tau\right]\Big\} \qquad (9)$$

$$w(x) = \int_0^x e^{-bt} I_0(ct)\,dt + l \int_0^x dt \int_0^t e^{-b\tau} I_0(c\tau)\,d\tau$$

$$\Psi(x, y) = -c_{-2}\frac{y}{\omega} + \left(-c_{-3}\frac{y}{\omega} + C_{-2}^2\frac{y^2}{\omega^2}\right)x + \ldots$$

$$a = \frac{k\rho_0}{U_0\rho_{10}\rho_{20}}\left(1 - \frac{\rho_{00}}{\rho_0}\right), \qquad b = \frac{k}{2U_0}\left(\frac{B_3}{B_2} + \frac{1}{B_1}\right)$$

$$c = \frac{k}{2U_0}\left(\frac{B_3}{B_2} - \frac{1}{B_1}\right), \qquad l = \frac{k}{U_0 B_1}, \qquad \omega^2 = \frac{B_1}{B_2}$$

$$m = \frac{k\rho_0}{U_0\rho_{10}\rho_{20}}, \qquad B_1 = \frac{\rho_{10}^2}{\rho_0^2}\rho_{20} + \frac{\rho_{20}^2}{\rho_{00}^2}\rho_{10}$$

$$t^* = x - \frac{y}{\omega}, \qquad B_2 = \frac{\rho_{10}^2}{\rho_0^2}\rho_{20}(M_1^2 - 1) + \frac{\rho_{20}^2}{\rho_{00}^2}\rho_{10}(M_2^2 - 1)$$

$$B_3 = \frac{\rho_{10}}{\rho_0}(M_1^2 - 1) + \frac{\rho_{20}}{\rho_{00}}(M_2^2 - 1) + \left(\frac{M_1^2}{\rho_0} - \frac{M_2^2}{\rho_{00}}\right)\left(\frac{\rho_{10}}{\rho_0}\rho_{20} - \frac{\rho_{20}}{\rho_{00}}\rho_{10}\right)$$

Here $I_0(cx)$ is the modified Bessel function.

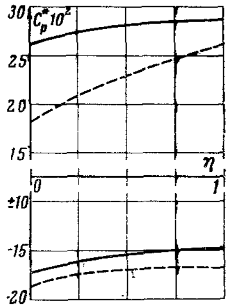

Fig. 2

To determine two-phase flow interaction with a slender body we must examine the motion of one phase in the region adjacent to the surface of the body. It is obvious that we have the equation for the disturbed flow velocity potential φ_2^*,

$$\varphi_{2xx}^* = \omega_0^2 \varphi_{2yy}^* \left(\omega_0^2 = \frac{1}{M_2^2 - 1}\right) \tag{10}$$

which has a solution of the form

$$\varphi_2^*(x, y) = \frac{1}{2}\left[f\left(x - \frac{y}{\omega_0}\right) + f\left(x + \frac{y}{\omega_0}\right)\right] + \frac{\omega_0}{2}\int_{\varkappa_-}^{\varkappa_+} F(z)\,dz \quad \left(\begin{matrix}\varkappa_+ = x + y/\omega_0 \\ \varkappa_- = x - y/\omega_0\end{matrix}\right) \tag{11}$$

Here $f(x)$ and $F(x)$ are functions which are known (as a result of the condition of continuity of the velocity of the light component on the phase interface) from the solution in the two-component-medium flow region. As we noted previously, at the surface of the slender body the following flow condition is satisfied:

$$\varphi_{2y}^*(x, y_0) = -U_0\beta_0(x)$$

Here $\beta_0(x)$ is the inclination of the tangents to the profile elements.

Substituting (11) into (12), we obtain the formula for finding the body shape, which is not presented because of its length. The pressure coefficient is

$$c_p^* = \frac{p - p_0}{\frac{1}{2}\rho_0 U_0^2} = -2\beta_0(\eta)\,\omega\left\{\frac{dw(\eta)}{d\eta} + \left(1 - \frac{\rho_{00}}{\rho_0}\right)\left[1 - \left(1 - \frac{\rho_{20}}{\rho_{00}}\right)\frac{\beta}{\beta_0(\eta)}\right] \times \frac{\partial}{\partial\eta}\int_0^\eta e^{-(\eta - t)} w(t)\,dt\right\} \tag{13}$$

Here $mx = \eta$, $my = \xi$ are dimensionless variables.

We note that in the absence of one of the media the Prandtl-Ackeret formula [9] follows from (13).

The compressive flow is studied similarly. Concrete calculations in the case

$$M_1 = 1.85,\ M_2 = 1.5,\ \rho_0 = 0.121\ \mathrm{kg \cdot sec^2/m^4}$$
$$\rho_{00} = 0.075\ \mathrm{kg \cdot sec^2/m^4},\ \rho_{10} = 0.1089\ \mathrm{kg \cdot sec^2/m^4}$$
$$\beta = 0.0875,\ K = 50\ \mathrm{kg \cdot sec/m^4}$$

show that the form of the slender body surface is a wedgelike profile (shaded region in Fig. 1), the pressure coefficient curve of the two-phase medium (solid curves in Fig. 2) is located above that for the single-phase medium (dashed curves in Fig. 2), and in the case of expansive (compressive) flow the light component of the medium acquires higher (lower) velocity than the heavy component. In Fig. 2 the upper two (solid and dashed) curves apply to compressive flow, the lower apply to expansive flow.

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